Math 236, solutions to practice exam

1. Here is a truth table involving the two propositions:

P	Q	R	$Q \vee R$	$P \Rightarrow (Q \vee R)$	$P \Rightarrow Q$	$P \Rightarrow R$	$(P \Rightarrow Q) \vee (P \Rightarrow R)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T
F	F	T	T	T	T	T	T
F	F	F	F	T	T	T	T

Comparing the fifth column and the last column, we see that they are the same, and hence the two propositions are logically equivalent.

2. There exists a positive integer n such that for all positive integers k, either k is not prime or $k^2 > n$.

3. We argue by induction on n.

When $n = 1$, the claim is that $\frac{1}{1 \cdot 5} = \frac{1}{4(1)+1}$, which is clearly true.

Now suppose that $k \geq 1$ is given, and that the result holds for k. Then our induction hypothesis says that

$$\frac{1}{1 \cdot 5} + \frac{1}{5 \cdot 9} + \cdots + \frac{1}{(4k-3)(4k+1)} = \frac{k}{4k+1}.$$

Now note that $\frac{1}{(4(k+1)-3)(4(k+1)+4)} = \frac{1}{(4k+1)(4k+5)}$, and when we add this to both sides of the equation above, we find

$$\begin{aligned}
\frac{1}{1 \cdot 5} + \frac{1}{5 \cdot 9} + \cdots + \frac{1}{(4k-3)(4k+1)} + \frac{1}{(4k+1)(4k+5)} &= \frac{k}{4k+1} + \frac{1}{(4k+1)(4k+5)} \\
&= \frac{k(4k+5)+1}{(4k+1)(4k+5)} \\
&= \frac{(4k+1)(k+1)}{(4k+1)(4k+5)} \\
&= \frac{k+1}{4k+5}
\end{aligned}$$

This chain of equations establishes the result for $k+1$, and completes the proof.

4. (a) This statement is false. Here is a counterexample: let $A = \{1\}$ and $B = \{2\}$. Then we have $\mathcal{P}(A \cup B) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$ and $\mathcal{P}(A) \cup \mathcal{P}(B) = \{\emptyset, \{1\}, \{2\}\}$. Note that $\{1,2\} \in \mathcal{P}(A \cup B)$ but $\{1,2\} \notin \mathcal{P}(A) \cup \mathcal{P}(B)$.

(b) This statement is true. To prove it, suppose that $S \in \mathcal{P}(A) \cup \mathcal{P}(B)$. Then either $S \in \mathcal{P}(A)$ or $S \in \mathcal{P}(B)$. If $S \in \mathcal{P}(A)$, then $S \subseteq A$, and therefore $S \subseteq A \cup B$, whence $S \in \mathcal{P}(A \cup B)$. Similarly, if $S \in \mathcal{P}(B)$, then $S \subseteq B$, and therefore $S \subseteq A \cup B$, whence $S \in \mathcal{P}(A \cup B)$. In either case, $S \in \mathcal{P}(A \cup B)$, so the result holds.

(c) This statement is false, since we proved in part (a) that $\mathcal{P}(A \cup B) \not\subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$.

5. To prove that a number is *not* rational, it's a good idea to use an indirect proof. Let's use a proof by contradiction. Suppose to the contrary that $\log_3(5)$ is rational. Then there exist integers p and q such that $\log_3(5) = p/q$. Note that since $\log_3(5)$ is positive (in fact, since $5 > 3$, it is at least 1), we in fact have that both p and q are positive. By the definition of logarithm, $\log_3(5) = p/q$ means that $3^{p/q} = 5$. Raising both sides to the q power then gives $3^p = 5^q$. But 3^p is divisible by 3 (because p is a positive integer), and 5^q is not. This contradiction proves the theorem.

6. (a) R is not symmetric, since $(2,1) \in R$ but $(1,2) \notin R$. R is not anti-symmetric, since $(1,3) \in R$ and $(3,1) \in R$ but $1 \neq 3$.

   (b) To be systematic about this, we need to first consider all $(x,y) \in R$ with $x = 1$. This gives just one pair, namely $(1,3)$, so $y = 3$. Now let's find all (y,z) with $y = 3$, namely $(3,1)$ and $(3,3)$. Transitive requires that since R contains the two pairs $(1,3)$ and $(3,1)$, then S must contain $(1,1)$. Also, since R contains $(1,3)$ and $(3,3)$, S must contain $(1,3)$; but R already has $(1,3)$, so we don't need to add anything.

   So we must add $(1,1)$ to R to get the relation $\{(1,1),(2,1),(1,3),(3,1),(3,3),(4,1)\}$. Considering all $(x,y) \in R$ with $x = 1$ as before, we see that we do not need to add any elements to the new relation. So consider all $(x,y) \in R$ with $x = 2$. The only one that needs attention is the pair $(2,1)$ and $(1,3)$; we must add $(2,3)$, which gives a new relation

   $$\{(1,1),(2,1),(2,3),(1,3),(3,1),(3,3),(4,1)\}$$

   Now if we considering all $(x,y) \in R$ with $x = 1$ or $x = 2$, we need to add no new elements. The same goes for $x = 3$. To account for the pairs with $x = 4$, we need to add $(4,3)$. Thus we take

   $$S = \{(1,1),(2,1),(2,3),(1,3),(3,1),(3,3),(4,1),(4,3)\}.$$

7. (a) Reflexive: Yes, since $(A - A) = \emptyset$, so $(A - A) \cup (A - A) = \emptyset$.

   (b) Symmetric: Yes, since $S \cup T = T \cup S$ for any sets S and T, and so $A\ R\ B$ implies $(A - B) \cup (B - A) = \emptyset$, which implies $(B - A) \cup (A - B) = \emptyset$, which implies $B\ R\ A$.

   (c) Anti-symmetric: Yes. Suppose that $A\ R\ B$ and $B\ R\ A$. Then $(A - B) \cup (B - A) = \emptyset$, which implies that both $(A - B)$ and $(B - A)$ are empty, for if either contained an element, then $(A - B) \cup (B - A) \neq \emptyset$. Now $(A - B)$ being empty implies that there does not exist x with $x \in A$ and $x \notin B$. Hence for all x, either $x \notin A$ or $x \in B$. Therefore if $x \in A$, then we must have $x \in B$. This proves that $A \subseteq B$. The same argument, with the roles of A and B reversed, shows that if $(B - A)$ is empty, then $B \subseteq A$. We've now shown that $(A - B) \cup (B - A) = \emptyset$ implies $A = B$. This proves that R is anti-symmetric.

   (d) Transitive: Yes. Suppose that $A\ R\ B$ and $B\ R\ C$. In part (c), we showed that this implies $A = B$ and $B = C$. Hence $A = C$, and so by part (a), $A\ R\ C$. Hence R is transitive.